

4

Overview In this chapter we discuss charge in motion, or electric *current*. The *current density* is defined as the current per cross-sectional area. It is related to the charge density by the *continuity equation*. In most cases, the current density is proportional to the electric field; the constant of proportionality is called the *conductivity*, with the inverse of the conductivity being the *resistivity*. *Ohm's law* gives an equivalent way of expressing this proportionality. We show in detail how the conductivity arises on a molecular level, by considering the drift velocity of the charge carriers when an electric field is applied. We then look at how this applies to *metals* and *semiconductors*. In a *circuit*, an *electromotive force (emf)* drives the current. A battery produces an emf by means of chemical reactions. The current in a circuit can be found either by reducing the circuit via the series and parallel rules for resistors, or by using Kirchhoff's rules. The *power* dissipated in a resistor depends on the resistance and the current passing through it. Any circuit can be reduced to a *Thévenin equivalent* circuit involving one resistor and one emf source. We end the chapter by investigating how the current changes in an *RC* circuit.

Electric currents

4.1 Electric current and current density

An electric current is charge in motion. The carriers of the charge can be physical particles like electrons or protons, which may or may not be attached to larger objects, atoms or molecules. Here we are not concerned with the nature of the charge carriers but only with the net transport of electric charge their motion causes. The electric current in a wire

is the amount of charge passing a fixed mark on the wire in unit time. The SI unit of current is the coulomb/second, which is called an *ampere* (amp, or A):

$$1 \text{ ampere} = 1 \frac{\text{coulomb}}{\text{second}}. \quad (4.1)$$

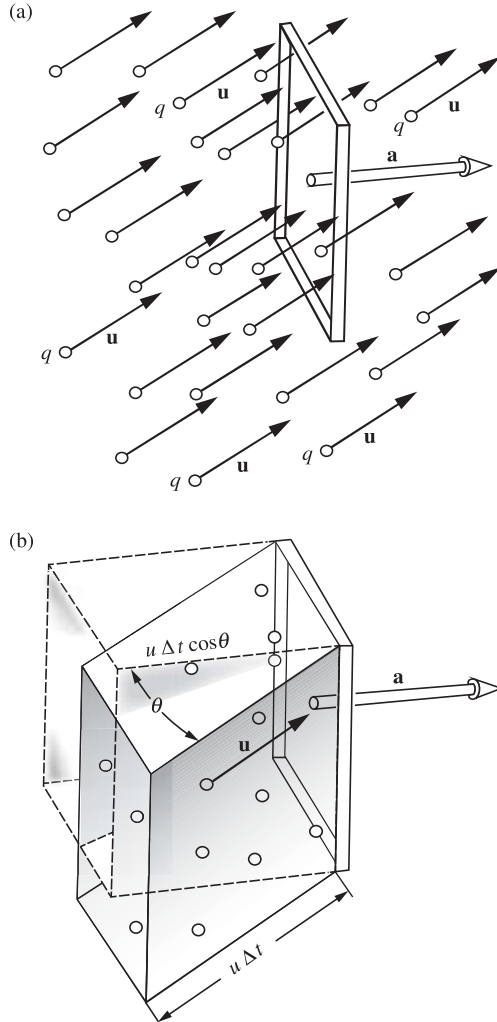


Figure 4.1.

(a) A swarm of charged particles all moving with the same velocity $\mathbf{u}$. The frame has area a . The particles that will pass through the frame in the next Δt seconds are those now contained in the oblique prism (b). The prism has base area a and altitude $u \Delta t \cos \theta$, hence its volume is $au \Delta t \cos \theta$ or $\mathbf{a} \cdot \mathbf{u} \Delta t$.

In Gaussian units current is expressed in esu/second. A current of 1 A is the same as a current of $2.998 \cdot 10^9$ esu/s, which is equivalent to $6.24 \cdot 10^{18}$ elementary electronic charges per second.

It is the net charge transport that counts, with due regard to sign. Negative charge moving east is equivalent to positive charge moving west. Water flowing through a hose could be said to involve the transport of an immense amount of charge – about $3 \cdot 10^{23}$ electrons per gram of water! But since an equal number of protons move along with the electrons (every water molecule contains ten of each), the electric current is zero. On the other hand, if you were to charge negatively a nylon thread and pull it steadily through a nonconducting tube, that would constitute an electric current, in the direction opposite to that of the motion of the thread.

We have been considering current along a well-defined path, like a wire. If the current is *steady* – that is, unchanging in time – it must be the same at every point along the wire, just as with steady traffic the same number of cars must pass, per hour, different points along an unbranching road.

A more general kind of current, or charge transport, involves charge carriers moving around in three-dimensional space. To describe this we need the concept of *current density*. We have to consider average quantities, for charge carriers are discrete particles. We must suppose, as we did in defining the charge density ρ , that our scale of distances is such that any small region we wish to average over contains very many particles of any class we are concerned with.

Consider first a special situation in which there are n particles per cubic meter, on the average, all moving with the same vector velocity $\mathbf{u}$ and carrying the same charge q . Imagine a small frame of area $\mathbf{a}$ fixed in some orientation, as in Fig. 4.1(a). How many particles pass through the frame in a time interval Δt ? If Δt begins the instant shown in Fig. 4.1(a) and (b), the particles destined to pass through the frame in the next Δt interval will be just those now located within the oblique prism in Fig. 4.1(b). This prism has the frame area as its base and an edge length $u \Delta t$, which is the distance any particle will travel in a time Δt . Particles outside this prism will either miss the window or fail to reach it. The volume of the prism is the product (base) $\times$ (altitude), or $au \Delta t \cos \theta$, which can be written $\mathbf{a} \cdot \mathbf{u} \Delta t$. On the average, the number of particles found in such a volume will be $n\mathbf{a} \cdot \mathbf{u} \Delta t$. Hence the average *rate* at which charge is

passing through the frame, that is, the current through the frame, which we shall call I_a , is

$$I_a = \frac{q(n\mathbf{a} \cdot \mathbf{u} \Delta t)}{\Delta t} = nq\mathbf{a} \cdot \mathbf{u}. \quad (4.2)$$

Suppose we had many classes of particles in the swarm, differing in charge q , in velocity vector $\mathbf{u}$, or in both. Each would make its own contribution to the current. Let us tag each kind by a subscript k . The k th class has charge q_k on each particle, moves with velocity vector $\mathbf{u}_k$, and is present with an average population density of n_k such particles per cubic meter. The resulting current through the frame is then

$$I_a = n_1 q_1 \mathbf{a} \cdot \mathbf{u}_1 + n_2 q_2 \mathbf{a} \cdot \mathbf{u}_2 + \cdots = \mathbf{a} \cdot \sum_k n_k q_k \mathbf{u}_k. \quad (4.3)$$

On the right is the scalar product of the vector $\mathbf{a}$ with a vector quantity that we shall call the current density $\mathbf{J}$:

$$\boxed{\mathbf{J} = \sum_k n_k q_k \mathbf{u}_k} \quad (4.4)$$

The SI unit of current density is amperes per square meter (A/m^2),¹ or equivalently coulombs per second per square meter ($\text{C s}^{-1} \text{m}^{-2}$), although technically the ampere is a fundamental SI unit while the coulomb is not (a coulomb is defined as one ampere-second). The Gaussian unit of current density is esu per second per square centimeter ($\text{esu s}^{-1} \text{cm}^{-2}$).

Let's look at the contribution to the current density $\mathbf{J}$ from one variety of charge carriers, electrons say, which may be present with many different velocities. In a typical conductor, the electrons will have an almost random distribution of velocities, varying widely in direction and magnitude. Let N_e be the total number of electrons per unit volume, of all velocities. We can divide the electrons into many groups, each of which contains electrons with nearly the same speed and direction. The *average velocity* of all the electrons, like any average, would then be calculated by summing over the groups, weighting each velocity by the number in the group, and dividing by the total number. That is,

$$\bar{\mathbf{u}} = \frac{1}{N_e} \sum_k n_k \mathbf{u}_k. \quad (4.5)$$

We use the bar over the top, as in $\bar{\mathbf{u}}$, to mean the average over a distribution. Comparing Eq. (4.5) with Eq. (4.4), we see that the contribution

¹ Sometimes one encounters current density expressed in A/cm^2 . Nothing is wrong with that; the meaning is perfectly clear as long as the units are stated. (Long before SI was promulgated, two or three generations of electrical engineers coped quite well with amperes per square inch!)

of the electrons to the current density can be written simply in terms of the average electron velocity. Remembering that the electron charge is $q = -e$, and using the subscript e to show that all quantities refer to this one type of charge carrier, we can write

$$\mathbf{J}_e = -eN_e\bar{\mathbf{u}}_e. \quad (4.6)$$

This may seem rather obvious, but we have gone through it step by step to make clear that the current through the frame depends only on the average velocity of the carriers, which often is only a tiny fraction, in magnitude, of their random speeds. Note that Eq. (4.6) can also be written as $\mathbf{J}_e = \rho_e\bar{\mathbf{u}}_e$, where $\rho_e = -eN_e$ is the volume charge density of the electrons.

4.2 Steady currents and charge conservation

The current I flowing through any surface S is just the surface integral

$$I = \int_S \mathbf{J} \cdot d\mathbf{a}. \quad (4.7)$$

We speak of a steady or stationary current system when the current density vector $\mathbf{J}$ remains constant in time everywhere. Steady currents have to obey the law of charge conservation. Consider some region of space completely enclosed by the balloonlike surface S . The surface integral of $\mathbf{J}$ over all of S gives the rate at which charge is leaving the volume enclosed. Now if charge forever pours out of, or into, a fixed volume, the charge density inside must grow infinite, unless some compensating charge is continually being created there. But charge creation is just what never happens. Therefore, for a truly time-independent current distribution, the surface integral of $\mathbf{J}$ over *any* closed surface must be zero. This is completely equivalent to the statement that, at every point in space,

$$\text{div } \mathbf{J} = 0. \quad (4.8)$$

To appreciate the equivalence, recall Gauss's theorem and our fundamental definition of divergence in terms of the surface integral over a small surface enclosing the location in question.

We can make a more general statement than Eq. (4.8). Suppose the current is not steady, $\mathbf{J}$ being a function of t as well as of x , y , and z . Then, since $\int_S \mathbf{J} \cdot d\mathbf{a}$ is the instantaneous rate at which charge is *leaving* the enclosed volume, while $\int_V \rho \, dv$ is the total charge *inside* the volume at any instant, we have

$$\int_S \mathbf{J} \cdot d\mathbf{a} = -\frac{d}{dt} \int_V \rho \, dv. \quad (4.9)$$

Letting the volume in question shrink down around any point (x, y, z) , the relation expressed in Eq. (4.9) becomes:²

$$\boxed{\operatorname{div} \mathbf{J} = -\frac{\partial \rho}{\partial t}} \quad (\text{time-dependent charge distribution}). \quad (4.10)$$

The time derivative of the charge density ρ is written as a partial derivative since ρ will usually be a function of spatial coordinates as well as time. Equations (4.9) and (4.10) express the (*local*) *conservation of charge*: no charge can flow away from a place without diminishing the amount of charge that is there. Equation (4.10) is known as the *continuity equation*.

Example (Vacuum diode) An instructive example of a stationary current distribution occurs in the plane diode, a two-electrode vacuum tube; see Fig. 4.2. One electrode, the cathode, is coated with a material that emits electrons copiously when heated. The other electrode, the anode, is simply a metal plate. By means of a battery the anode is maintained at a positive potential with respect to the cathode. Electrons emerge from this hot cathode with very low velocities and then, being negatively charged, are accelerated toward the positive anode by the electric field between cathode and anode. In the space between the cathode and anode the electric current consists of these moving electrons. The circuit is completed by the flow of electrons in external wires, possibly by the movement of ions in a battery, and so on, with which we are not here concerned.

In this diode the local density of charge in any region, ρ , is simply $-ne$, where n is the local density of electrons, in electrons per cubic meter. The local current density $\mathbf{J}$ is $\rho\mathbf{v}$, where $\mathbf{v}$ is the velocity of electrons in that region. In the plane-parallel diode we may assume $\mathbf{J}$ has no y or z components. If conditions are steady, it follows then that J_x must be independent of x , for if $\operatorname{div} \mathbf{J} = 0$ as Eq. (4.8) says, $\partial J_x / \partial x$ must be zero if $J_y = J_z = 0$. This is belaboring the obvious; if we have a steady stream of electrons moving in the x direction only, the same number per second have to cross any intermediate plane between cathode and anode. We conclude that ρv is constant. But observe that v is *not* constant; it varies with x because the electrons are accelerated by the field. Hence ρ is not constant either. Instead, the negative charge density is high near the cathode and low near the anode, just as the density of cars on an expressway is high near a traffic slowdown and low where traffic is moving at high speed.

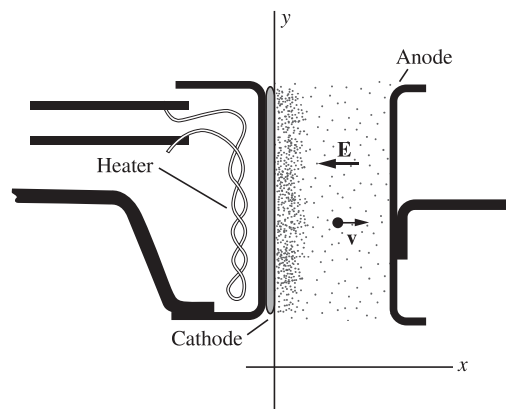


Figure 4.2. A vacuum diode with plane-parallel cathode and anode.

4.3 Electrical conductivity and Ohm's law

There are many ways of causing charge to move, including what we might call “bodily transport” of the charge carriers. In the Van de Graaff

² If the step between Eqs. (4.9) and (4.10) is not obvious, look back at our fundamental definition of divergence in Chapter 2. As the volume shrinks, we can eventually take ρ outside the volume integral on the right. The volume integral is to be carried out at one instant of time. The time derivative thus depends on the difference between $\rho \int dv$ at t and at $t + dt$. The only difference is due to the change of ρ there, since the boundary of the volume remains in the same place.

electrostatic generator (see Problem 4.1) an insulating belt is given a surface charge, which it conveys to another electrode for removal, much as an escalator conveys people. That constitutes a perfectly good current. In the atmosphere, charged water droplets falling because of their weight form a component of the electric current system of the earth. In this section we shall be interested in a more common agent of charge transport, the force exerted on a charge carrier by an electric field. An electric field $\mathbf{E}$ pushes positive charge carriers in one direction, negative charge carriers in the opposite direction. If either or both can move, the result is an electric current in the direction of $\mathbf{E}$. In most substances, and over a wide range of electric field strengths, we find that the current density is proportional to the strength of the electric field that causes it. The linear relation between current density and field is expressed by

$$\mathbf{J} = \sigma \mathbf{E} \quad (4.11)$$

The factor σ is called the *conductivity* of the material. Its value depends on the material in question; it is very large for metallic conductors, extremely small for good insulators. It may depend too on the physical state of the material – on its temperature, for instance. But with such conditions given, it does not depend on the magnitude of $\mathbf{E}$. If you double the field strength, holding everything else constant, you get twice the current density.

After everything we said in Chapter 3 about the electric field being zero inside a conductor, you might be wondering why we are now talking about a nonzero internal field. The reason is that in Chapter 3 we were dealing with static situations, that is, ones in which all the charges have settled down after some initial motion. In such a setup, the charges pile up at certain locations and create a field that internally cancels an applied field. But when dealing with currents in conductors, we are not letting the charges pile up, which means that things can't settle down. For example, a battery feeds in electrons at one end of a wire and takes them out at the other end. If the electrons were *not* taken out at the other end, then they *would* pile up there, and the electric field would eventually (actually very quickly) become zero inside.

The units of σ are the units of $\mathbf{J}$ (namely $\text{C s}^{-1} \text{m}^{-2}$) divided by the units of $\mathbf{E}$ (namely V/m or N/C). You can quickly show that this yields $\text{C}^2 \text{s kg}^{-1} \text{m}^{-3}$. However, it is customary to write the units of σ as the reciprocal of ohm-meter, $(\text{ohm-m})^{-1}$, where the ohm, which is the unit of resistance, is defined below.

In Eq. (4.11), σ may be considered a scalar quantity, implying that the direction of $\mathbf{J}$ is always the same as the direction of $\mathbf{E}$. That is surely what we would expect within a material whose structure has no “built-in” preferred direction. Materials do exist in which the electrical conductivity itself depends on the angle the applied field $\mathbf{E}$ makes with

some intrinsic axis in the material. One example is a single crystal of graphite, which has a layered structure on an atomic scale. For another example, see Problem 4.5. In such cases $\mathbf{J}$ may not have the direction of $\mathbf{E}$. But there still are linear relations between the components of $\mathbf{J}$ and the components of $\mathbf{E}$, relations expressed by Eq. (4.11) with σ a *tensor* quantity instead of a scalar.³ From now on we'll consider only *isotropic* materials, those within which the electrical conductivity is the same in all directions.

Equation (4.11) is a statement of *Ohm's law*. It is an *empirical* law, a generalization derived from experiment, not a theorem that must be universally obeyed. In fact, Ohm's law is bound to fail in the case of any particular material if the electric field is too strong. And we shall meet some interesting and useful materials in which “nonohmic” behavior occurs in rather weak fields. Nevertheless, the remarkable fact is the enormous range over which, in the large majority of materials, current density is proportional to electric field. Later in this chapter we'll explain why this should be so. But now, taking Eq. (4.11) for granted, we want to work out its consequences. We are interested in the total current I flowing through a wire or a conductor of any other shape with well-defined ends, or terminals, and in the difference in potential between those terminals, for which we'll use the symbol V (for *voltage*) rather than $\phi_1 - \phi_2$ or ϕ_{12} .

Now, I is the surface integral of $\mathbf{J}$ over a cross section of the conductor, which implies that I is proportional to $\mathbf{J}$. Also, V is the line integral of $\mathbf{E}$ on a path through the conductor from one terminal to the other, which implies that V is proportional to $\mathbf{E}$. Therefore, if $\mathbf{J}$ is proportional to $\mathbf{E}$ everywhere inside a conductor as Eq. (4.11) states, then I must be proportional to V . The relation of V to I is therefore another expression of Ohm's law, which we'll write this way:

$$\boxed{V = IR} \quad (\text{Ohm's law}). \quad (4.12)$$

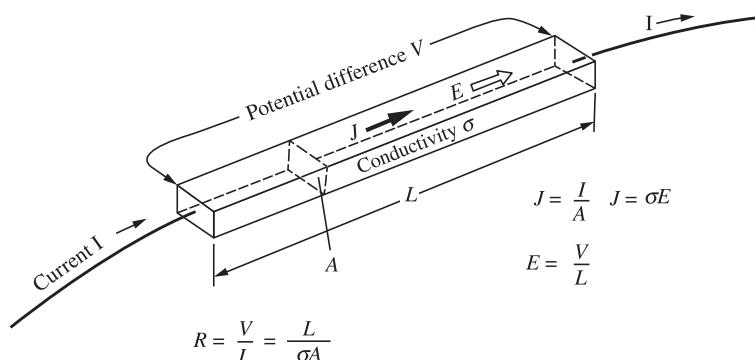
The constant R is the *resistance* of the conductor between the two terminals; R depends on the size and shape of the conductor and the

³ The most general linear relation between the two vectors $\mathbf{J}$ and $\mathbf{E}$ would be expressed as follows. In place of the three equations equivalent to Eq. (4.11), namely, $J_x = \sigma E_x$, $J_y = \sigma E_y$, and $J_z = \sigma E_z$, we would have $J_x = \sigma_{xx}E_x + \sigma_{xy}E_y + \sigma_{xz}E_z$, $J_y = \sigma_{yx}E_x + \sigma_{yy}E_y + \sigma_{yz}E_z$, and $J_z = \sigma_{zx}E_x + \sigma_{zy}E_y + \sigma_{zz}E_z$. These relations can be compactly summarized in the matrix equation,

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

The nine coefficients σ_{xx} , σ_{xy} , etc., make up a *tensor*, which here is just a matrix. In this case, because of a symmetry requirement, it would turn out that $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$, $\sigma_{xz} = \sigma_{zx}$. Furthermore, by a suitable orientation of the x , y , z axes, all the coefficients could be rendered zero except σ_{xx} , σ_{yy} , and σ_{zz} .

Figure 4.3.
The resistance of a conductor of length L , uniform cross-sectional area A , and conductivity σ .



conductivity σ of the material. The simplest example is a solid rod of cross-sectional area A and length L . A steady current I flows through this rod from one end to the other (Fig. 4.3). Of course there must be conductors to carry the current to and from the rod. We consider the terminals of the rod to be the points where these conductors are attached. Inside the rod the current density is given by

$$J = \frac{I}{A}, \quad (4.13)$$

and the electric field strength is given by

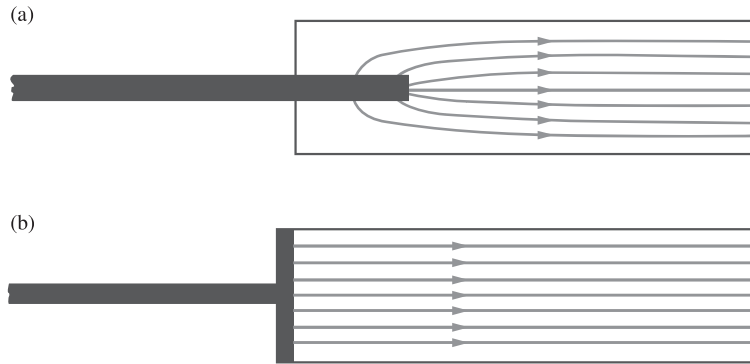
$$E = \frac{V}{L}. \quad (4.14)$$

The resistance R in Eq. (4.12) is V/I . Using Eqs. (4.11), (4.13), and (4.14) we easily find that

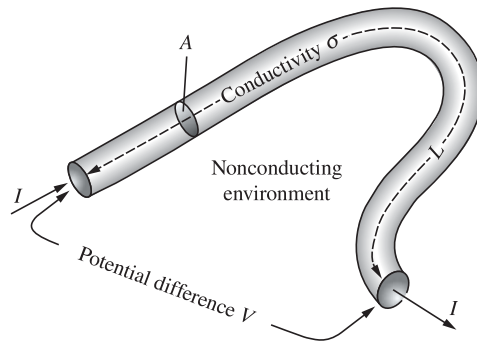
$$R = \frac{V}{I} = \frac{LE}{AJ} = \frac{L}{A\sigma}. \quad (4.15)$$

On the way to this simple formula we made some tacit assumptions. First, we assumed the current density is uniform over the cross section of the bar. To see why that must be so, imagine that J is actually greater along one side of the bar than on the other. Then E must also be greater along that side. But then the line integral of $\mathbf{E}$ from one terminal to the other would be greater for a path along one side than for a path along the other, and that cannot be true for an electrostatic field.

A second assumption was that $\mathbf{J}$ kept its uniform magnitude and direction right out to the ends of the bar. Whether that is true or not depends on the external conductors that carry current to and from the bar and how they are attached. Compare Fig. 4.4(a) with Fig. 4.4(b). Suppose that the terminal in (b) is made of material with a conductivity much higher than that of the bar. That will make the plane of the end of the bar an equipotential surface, creating the current system to which Eq. (4.15) applies *exactly*. But all we can say in general about such “end effects” is that Eq. (4.15) will give R to a good approximation if the width of the bar is small compared with its length.

**Figure 4.4.**

Different ways in which the current I might be introduced into the conducting bar. In (a) it has to spread out before the current density $\mathbf{J}$ becomes uniform. In (b) if the external conductor has much higher conductivity than the bar, the end of the bar will be an equipotential and the current density will be uniform from the beginning. For long thin conductors, such as ordinary wires, the difference is negligible.



$$R = \frac{L}{\sigma A}$$

Figure 4.5.

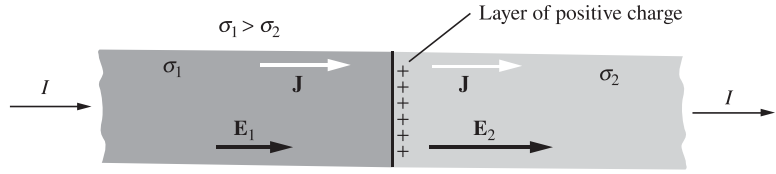
As long as our conductors are surrounded by a nonconducting medium (air, oil, vacuum, etc.), the resistance R between the terminals doesn't depend on the shape, only on the length of the conductor and its cross-sectional area.

A third assumption is that the bar is surrounded by an electrically nonconducting medium. Without that, we could not even *define* an isolated current path with terminals and talk about *the* current I and the resistance R . In other words, it is the enormous difference in conductivity between good insulators, including air, and conductors that makes *wires*, as we know them, possible. Imagine the conducting rod of Fig. 4.3 bent into some other shape, as in Fig. 4.5. Because it is embedded in a nonconducting medium into which current cannot leak, the problem presented in Fig. 4.5 is for all practical purposes the same as the one in Fig. 4.3 which we have already solved. Equation (4.15) applies to a bent wire as well as a straight rod, if we measure L along the wire.

In a region where the conductivity σ is constant, the steady current condition $\text{div } \mathbf{J} = 0$ (Eq. (4.8)) together with Eq. (4.11) implies that $\text{div } \mathbf{E} = 0$ also. This tells us that the charge density is zero within that region. On the other hand, if σ varies from one place to another in the conducting medium, steady current flow may entail the presence of static charge within the conductor. Figure 4.6 shows a simple example, a bar made of two materials of different conductivity, σ_1 and σ_2 . The current density $\mathbf{J}$ must be the same on the two sides of the interface; otherwise charge would continue to pile up there. It follows that the electric field $\mathbf{E}$

Figure 4.6.

When current flows through this composite conductor, a layer of static charge appears at the interface between the two materials, so as to provide the necessary jump in the electric field $\mathbf{E}$. In this example $\sigma_2 < \sigma_1$, hence E_2 must be greater than E_1 .



must be different in the two regions, with an abrupt jump in value at the interface. As Gauss's law tells us, such a discontinuity in $\mathbf{E}$ must reflect the presence of a layer of static charge at the interface. Problem 4.2 looks further into this example.

Instead of the conductivity σ we could have used its reciprocal, the *resistivity* ρ , in stating the relation between electric field and current density:

$$\mathbf{J} = \left(\frac{1}{\rho} \right) \mathbf{E}. \quad (4.16)$$

It is customary to use ρ as the symbol for resistivity and σ as the symbol for conductivity in spite of their use in some of our other equations for volume charge density and surface charge density. In the rest of this chapter ρ will always denote resistivity and σ conductivity. Equation (4.15) written in terms of resistivity becomes

$$\boxed{R = \frac{\rho L}{A}} \quad (4.17)$$

The SI unit for resistance is defined to be the *ohm* (denoted by Ω), which is given by Eq. (4.12) as

$$1 \text{ ohm} = 1 \frac{\text{volt}}{\text{ampere}}. \quad (4.18)$$

In terms of other SI units, you can show that 1 ohm equals $1 \text{ kg m}^2 \text{ C}^{-2} \text{ s}^{-1}$. If resistance R is in ohms, it is evident from Eq. (4.17) that the resistivity ρ must have units of (ohms) $\times$ (meters). The official SI unit for ρ is therefore the ohm-meter. But another unit of length can be used with perfectly clear meaning. A unit commonly used for resistivity, in both the physics and technology of electrical conduction, is the ohm-centimeter (ohm-cm). If one chooses to measure resistivity in ohm-cm, the corresponding unit for conductivity is written as $\text{ohm}^{-1} \text{ cm}^{-1}$, or $(\text{ohm-cm})^{-1}$, and called “reciprocal ohm-cm.” It should be emphasized that Eqs. (4.11) through (4.17) are valid for any self-consistent choice of units.

Example (Lengthening a wire) A wire of pure tin is drawn through a die, reducing its diameter by 25 percent and increasing its length. By what factor is its resistance increased? It is then flattened into a ribbon by rolling, which results in a further increase in its length, now twice the original length. What is the overall change in resistance? Assume the density and resistivity remain constant throughout.

Solution Let A be the cross-sectional area, and let L be the length. The volume AL is constant, so $L \propto 1/A$. The resistance $R = \rho L/A$ is therefore proportional to $1/A^2$. If the die reduces the diameter by the factor $3/4$, then it reduces A by the factor $(3/4)^2$. The resistance is therefore multiplied by the factor $1/(3/4)^4 = 3.16$. In terms of the radius r , the resistance is proportional to $1/r^4$.

Since $A \propto 1/L$, we can alternatively say that the resistance $R = \rho L/A$ is proportional to L^2 . An overall increase in L by the factor 2 therefore yields an overall increase in R by the factor $2^2 = 4$.

In Gaussian units, the unit of charge can be expressed in terms of other fundamental units, because Coulomb's law with a dimensionless coefficient yields $1 \text{ esu} = 1 \text{ g}^{1/2} \text{ cm}^{3/2} \text{ s}^{-1}$, as you can verify. You can use this to show that the units of resistance are s/cm. Since Eq. (4.17) still tells us that the resistivity ρ has dimensions of (resistance) $\times$ (length), we see that the Gaussian unit of ρ is simply the second. The analogous statement in the SI system, as you can check, is that the units of ρ are seconds divided by the units of ϵ_0 . Hence $\epsilon_0 \rho$ has the dimensions of time. This association of a resistivity with a time has a natural interpretation which will be explained in Section 4.11.

The conductivity and resistivity of a few materials are given in different units for comparison in Table 4.1. The key conversion factor is also given (see Appendix C for the derivation).

Example (Drift velocity in a copper wire) A copper wire $L = 1 \text{ km}$ long is connected across a $V = 6 \text{ V}$ battery. The resistivity of the copper is $\rho = 1.7 \cdot 10^{-8} \text{ ohm-meter}$, and the number of conduction electrons per cubic meter is $N = 8 \cdot 10^{28} \text{ m}^{-3}$. What is the drift velocity of the conduction electrons under these circumstances? How long does it take an electron to drift once around the circuit?

Solution Equation (4.6) gives the magnitude of the current density as $J = Nev$, so the drift velocity is $v = J/Ne$. But J is given by $J = \sigma E = (1/\rho)(V/L)$. Substituting this into $v = J/Ne$ yields

$$\begin{aligned} v &= \frac{V}{\rho L N e} = \frac{6 \text{ V}}{(1.7 \cdot 10^{-8} \text{ ohm-m})(1000 \text{ m})(8 \cdot 10^{28} \text{ m}^{-3})(1.6 \cdot 10^{-19} \text{ C})} \\ &= 2.8 \cdot 10^{-5} \text{ m/s.} \end{aligned} \quad (4.19)$$

This is *much* slower than the average thermal speed of an electron at room temperature, which happens to be about 10^5 m/s . The time to drift once around the

Table 4.1.
Resistivity and its reciprocal, conductivity, for a few materials

Material	Resistivity ρ	Conductivity σ
Pure copper, 273 K	$1.56 \cdot 10^{-8}$ ohm-m $1.73 \cdot 10^{-18}$ s	$6.4 \cdot 10^7$ (ohm-m) $^{-1}$ $5.8 \cdot 10^{17}$ s $^{-1}$
Pure copper, 373 K	$2.24 \cdot 10^{-8}$ ohm-m $2.47 \cdot 10^{-18}$ s	$4.5 \cdot 10^7$ (ohm-m) $^{-1}$ $4.0 \cdot 10^{17}$ s $^{-1}$
Pure germanium, 273 K	2 ohm-m $2.2 \cdot 10^{-10}$ s	0.5 (ohm-m) $^{-1}$ $4.5 \cdot 10^9$ s $^{-1}$
Pure germanium, 500 K	$1.2 \cdot 10^{-3}$ ohm-m $1.3 \cdot 10^{-13}$ s	830 (ohm-m) $^{-1}$ $7.7 \cdot 10^{12}$ s $^{-1}$
Pure water, 291 K	$2.5 \cdot 10^5$ ohm-m $2.8 \cdot 10^{-5}$ s	$4.0 \cdot 10^{-6}$ (ohm-m) $^{-1}$ $3.6 \cdot 10^4$ s $^{-1}$
Seawater (varies with salinity)	0.25 ohm-m $2.8 \cdot 10^{-11}$ s	4 (ohm-m) $^{-1}$ $3.6 \cdot 10^{10}$ s $^{-1}$

Note: 1 ohm-meter = $1.11 \cdot 10^{-10}$ s.

circuit is $t = (1000 \text{ m}) / (2.8 \cdot 10^{-5} \text{ m/s}) = 3.6 \cdot 10^7 \text{ s}$, which is a little over a year. Note that v is independent of the cross-sectional area. This makes sense, because if we have two separate identical wires connected to the same voltage source, they have the same v . If we combine the two wires into one thicker wire, this shouldn't change the v .

When dealing with currents in wires, we generally assume that the wire is neutral. That is, we assume that the moving electrons have the same density per unit length as the stationary protons in the lattice. We should mention, however, that an actual current-carrying wire is *not* neutral. There are surface charges on the wire, as explained by [Marcus \(1941\)](#) and demonstrated by [Jefimenko \(1962\)](#). These charges are necessary for three reasons.

First, the surface charges keep the current flowing along the path of the wire. Consider a battery connected to a long wire, and let's say we put a bend in the wire far from the battery. If we then bend the wire in some other arbitrary manner, the battery doesn't "know" that we changed the shape, so it certainly can't be the cause of the electrons taking a new path through space. The cause of the new path must be the electric field due to nearby charges. These charges appear as surface charges on the wire.

Second, the existence of a net charge on the wire is necessary to create the proper flow of energy associated with the current. To get a

handle on this energy flow, we will have to wait until we learn about magnetic fields in Chapter 6 and the *Poynting vector* in Chapter 9. But for now we'll just say that to have the proper energy flow, there must be a component of the electric field pointing radially away from the wire. This component wouldn't exist if the net charge on the wire were zero.

Third, the surface charge causes the potential to change along the wire in a manner consistent with Ohm's law. See Jackson (1996) for more discussion on these three roles that the surface charges play.

However, having said all this, it turns out that in most of our discussions of circuits and currents in this book, we won't be interested in the electric field external to the wires. So we can generally ignore the surface charges, with no ill effects.

4.4 The physics of electrical conduction

4.4.1 Currents and ions

To explain electrical conduction we have to talk first about atoms and molecules. Remember that a neutral atom, one that contains as many electrons as there are protons in its nucleus, is *precisely* neutral (see Section 1.3). On such an object the net force exerted by an electric field is exactly zero. And even if the neutral atom were moved along by some other means, that would not be an electric current. The same holds for neutral molecules. Matter that consists only of neutral molecules ought to have zero electrical conductivity. Here one qualification is in order: we are concerned now with steady electric currents, that is, *direct* currents, not alternating currents. An alternating electric field could cause periodic deformation of a molecule, and that displacement of electric charge would be a true alternating electric current. We shall return to that subject in Chapter 10. For a steady current we need mobile charge carriers, or *ions*. These must be present in the material before the electric field is applied, for the electric fields we shall consider are not nearly strong enough to create ions by tearing electrons off molecules. Thus the physics of electrical conduction centers on two questions: how many ions are there in a unit volume of material, and how do these ions move in the presence of an electric field?

In pure water at room temperature approximately two H_2O molecules in a billion are, at any given moment, dissociated into negative ions, OH^- , and positive ions, H^+ . (Actually the positive ion is better described as OH_3^+ , that is, a proton attached to a water molecule.) This provides approximately $6 \cdot 10^{13}$ negative ions and an equal number of positive ions in a cubic centimeter of water.⁴ The motion of these ions in the

⁴ Students of chemistry may recall that the concentration of hydrogen ions in pure water corresponds to a pH value of 7.0, which means the concentration is $10^{-7.0}$ mole/liter. That is equivalent to $10^{-10.0}$ mole/cm³. A mole of anything is $6.02 \cdot 10^{23}$ things – hence the number $6 \cdot 10^{13}$ given above.

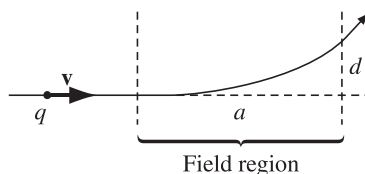


FIGURE 5.8

Problem 5.2 Find and sketch the trajectory of the particle in Ex. 5.2, if it starts at the origin with velocity

- (a) $\mathbf{v}(0) = (E/B)\hat{\mathbf{y}}$,
- (b) $\mathbf{v}(0) = (E/2B)\hat{\mathbf{y}}$,
- (c) $\mathbf{v}(0) = (E/B)(\hat{\mathbf{y}} + \hat{\mathbf{z}})$.

Problem 5.3 In 1897, J. J. Thomson “discovered” the electron by measuring the charge-to-mass ratio of “cathode rays” (actually, streams of electrons, with charge q and mass m) as follows:

- (a) First he passed the beam through uniform crossed electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ (mutually perpendicular, and both of them perpendicular to the beam), and adjusted the electric field until he got zero deflection. What, then, was the speed of the particles (in terms of E and B)?
- (b) Then he turned off the electric field, and measured the radius of curvature, R , of the beam, as deflected by the magnetic field alone. In terms of E , B , and R , what is the charge-to-mass ratio (q/m) of the particles?

5.1.3 ■ Currents

The **current** in a wire is the *charge per unit time* passing a given point. By definition, negative charges moving to the left count the same as positive ones to the right. This conveniently reflects the *physical* fact that almost all phenomena involving moving charges depend on the *product* of charge and velocity—if you reverse the signs of q and $\mathbf{v}$, you get the same answer, so it doesn’t really matter which you have. (The Lorentz force law is a case in point; the Hall effect (Prob. 5.41) is a notorious exception.) In practice, it is ordinarily the negatively charged electrons that do the moving—in the direction *opposite* to the electric current. To avoid the petty complications this entails, I shall often pretend it’s the positive charges that move, as in fact everyone assumed they did for a century or so after Benjamin Franklin established his unfortunate convention.⁵ Current is measured in coulombs-per-second, or **amperes** (A):

$$1 \text{ A} = 1 \text{ C/s.} \quad (5.12)$$

⁵If we called the electron plus and the proton minus, the problem would never arise. In the context of Franklin’s experiments with cat’s fur and glass rods, the choice was completely arbitrary.

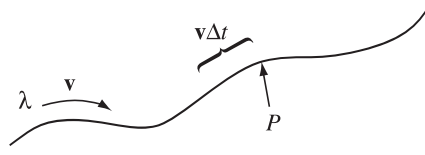


FIGURE 5.9

A line charge λ traveling down a wire at speed v (Fig. 5.9) constitutes a current

$$I = \lambda v, \quad (5.13)$$

because a segment of length $v\Delta t$, carrying charge $\lambda v\Delta t$, passes point P in a time interval Δt . Current is actually a *vector*:

$$\mathbf{I} = \lambda \mathbf{v}. \quad (5.14)$$

Because the path of the flow is dictated by the shape of the wire, one doesn't ordinarily bother to display the direction of $\mathbf{I}$ explicitly,⁶ but when it comes to surface and volume currents we cannot afford to be so casual, and for the sake of notational consistency it is a good idea to acknowledge the vectorial character of currents right from the start. A neutral wire, of course, contains as many stationary positive charges as mobile negative ones. The former do not contribute to the current—the charge density λ in Eq. 5.13 refers only to the *moving* charges. In the unusual situation where *both* types move, $\mathbf{I} = \lambda_+ \mathbf{v}_+ + \lambda_- \mathbf{v}_-$.

The magnetic force on a segment of current-carrying wire is

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \lambda dl = \int (\mathbf{I} \times \mathbf{B}) dl. \quad (5.15)$$

Inasmuch as $\mathbf{I}$ and $d\mathbf{l}$ both point in the same direction, we can just as well write this as

$$\mathbf{F}_{\text{mag}} = \int I (d\mathbf{l} \times \mathbf{B}). \quad (5.16)$$

Typically, the current is constant (in magnitude) along the wire, and in that case I comes outside the integral:

$$\mathbf{F}_{\text{mag}} = I \int (d\mathbf{l} \times \mathbf{B}). \quad (5.17)$$

Example 5.3. A rectangular loop of wire, supporting a mass m , hangs vertically with one end in a uniform magnetic field $\mathbf{B}$, which points into the page in the shaded region of Fig. 5.10. For what current I , in the loop, would the magnetic force upward exactly balance the gravitational force downward?

⁶For the same reason, if you are describing a locomotive constrained to move along a specified track, you would probably speak of its *speed*, rather than its velocity.

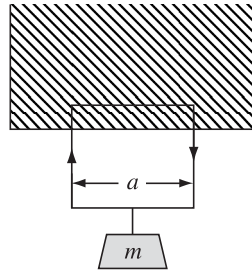


FIGURE 5.10

Solution

First of all, the current must circulate clockwise, in order for $(\mathbf{I} \times \mathbf{B})$ in the horizontal segment to point upward. The force is

$$F_{\text{mag}} = I B a,$$

where a is the width of the loop. (The magnetic forces on the two vertical segments cancel.) For F_{mag} to balance the weight (mg), we must therefore have

$$I = \frac{mg}{Ba}. \quad (5.18)$$

The weight just *hangs* there, suspended in mid-air!

What happens if we now *increase* the current? Then the upward magnetic force *exceeds* the downward force of gravity, and the loop rises, lifting the weight. *Somebody's* doing work, and it sure looks as though the magnetic force is responsible. Indeed, one is tempted to write

$$W_{\text{mag}} = F_{\text{mag}} h = I B a h, \quad (5.19)$$

where h is the distance the loop rises. But we know that magnetic forces *never* do work. What's going on here?

Well, when the loop starts to rise, the charges in the wire are no longer moving horizontally—their velocity now acquires an upward component u , the speed of the loop (Fig. 5.11), in addition to the horizontal component w associated with the current ($I = \lambda w$). The magnetic force, which is always perpendicular to the velocity, no longer points straight up, but tilts back. It is perpendicular to the *net* displacement of the charge (which is in the direction of $\mathbf{v}$), and therefore *it does no work on q* . It does have a vertical *component* (qwB); indeed, the net vertical force on all the charge (λa) in the upper segment of the loop is

$$F_{\text{vert}} = \lambda a w B = I B a \quad (5.20)$$

(as before); but now it also has a *horizontal* component (quB), which opposes the flow of current. Whoever is in charge of maintaining that current, therefore, must now *push* those charges along, against the backward component of the magnetic force.

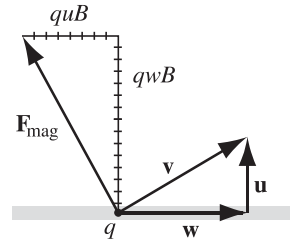


FIGURE 5.11

The total horizontal force on the top segment is

$$F_{\text{horiz}} = \lambda a u B. \quad (5.21)$$

In a time dt , the charges move a (horizontal) distance $w dt$, so the work done by this agency (presumably a battery or a generator) is

$$W_{\text{battery}} = \lambda a B \int u w dt = I B a h,$$

which is precisely what we naïvely attributed to the *magnetic* force in Eq. 5.19. Was work done in this process? Absolutely! Who did it? The battery! What, then, was the role of the magnetic force? Well, it *redirected* the horizontal force of the battery into the *vertical* motion of the loop and the weight.⁷

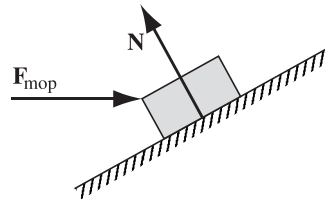


FIGURE 5.12

It may help to consider a mechanical analogy. Imagine you're sliding a trunk up a frictionless ramp, by pushing on it horizontally with a mop (Fig. 5.12). The normal force ($\mathbf{N}$) does no work, because it is perpendicular to the displacement. But it *does* have a vertical component (which in fact is what lifts the trunk), and a (backward) horizontal component (which you have to overcome by pushing on the mop). Who is doing the work here? *You* are, obviously—and yet your *force* (which is purely horizontal) is not (at least, not directly) what lifts the box. The

⁷If you like, the *vertical* component of $\mathbf{F}_{\text{mag}}$ does work lifting the car, but the *horizontal* component does equal *negative* work opposing the current. However you look at it, the *net* work done by the magnetic force is *zero*.

normal force plays the same passive (but crucial) role as the magnetic force in Ex. 5.3: while doing no work itself, it *redirects* the efforts of the active agent (you, or the battery, as the case may be), from horizontal to vertical.

When charge flows over a *surface*, we describe it by the **surface current density**, $\mathbf{K}$, defined as follows: Consider a “ribbon” of infinitesimal width $dl_{\perp}$, running parallel to the flow (Fig. 5.13). If the current in this ribbon is $d\mathbf{I}$, the surface current density is

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_{\perp}}. \quad (5.22)$$

In words, K is the *current per unit width*. In particular, if the (mobile) surface charge density is σ and its velocity is $\mathbf{v}$, then

$$\mathbf{K} = \sigma \mathbf{v}. \quad (5.23)$$

In general, $\mathbf{K}$ will vary from point to point over the surface, reflecting variations in σ and/or $\mathbf{v}$. The magnetic force on the surface current is

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) \sigma \, da = \int (\mathbf{K} \times \mathbf{B}) \, da. \quad (5.24)$$

Caveat: Just as $\mathbf{E}$ suffers a discontinuity at a surface *charge*, so $\mathbf{B}$ is discontinuous at a surface *current*. In Eq. 5.24, you must be careful to use the *average* field, just as we did in Sect. 2.5.3.

When the flow of charge is distributed throughout a three-dimensional region, we describe it by the **volume current density**, $\mathbf{J}$, defined as follows: Consider a “tube” of infinitesimal cross section $da_{\perp}$, running parallel to the flow (Fig. 5.14). If the current in this tube is $d\mathbf{I}$, the volume current density is

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}}. \quad (5.25)$$

In words, J is the *current per unit area*. If the (mobile) volume charge density is ρ and the velocity is $\mathbf{v}$, then

$$\mathbf{J} = \rho \mathbf{v}. \quad (5.26)$$

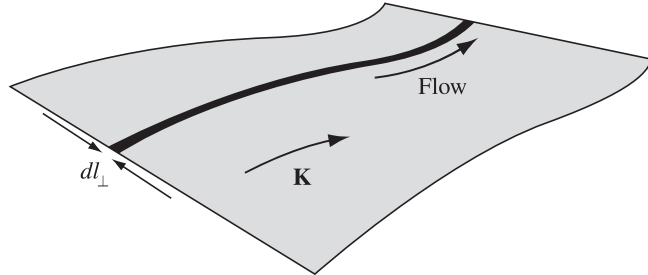


FIGURE 5.13

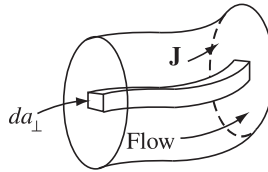


FIGURE 5.14

The magnetic force on a volume current is therefore

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) \rho d\tau = \int (\mathbf{J} \times \mathbf{B}) d\tau. \quad (5.27)$$

Example 5.4.

(a) A current I is uniformly distributed over a wire of circular cross section, with radius a (Fig. 5.15). Find the volume current density J .

Solution

The area (perpendicular to the flow) is πa^2 , so

$$J = \frac{I}{\pi a^2}.$$

This was trivial because the current density was uniform.

(b) Suppose the current density in the wire is proportional to the distance from the axis,

$$J = ks$$

(for some constant k). Find the total current in the wire.

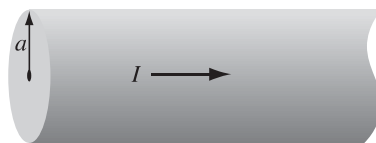


FIGURE 5.15

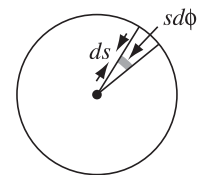


FIGURE 5.16

Solution

Because J varies with s , we must *integrate* Eq. 5.25. The current through the shaded patch (Fig. 5.16) is $J da_{\perp}$, and $da_{\perp} = s ds d\phi$. So

$$I = \int (ks)(s ds d\phi) = 2\pi k \int_0^a s^2 ds = \frac{2\pi k a^3}{3}.$$

According to Eq. 5.25, the total current crossing a surface S can be written as

$$I = \int_S J da_{\perp} = \int_S \mathbf{J} \cdot d\mathbf{a}. \quad (5.28)$$

(The dot product serves neatly to pick out the appropriate component of $d\mathbf{a}$.) In particular, the charge per unit time leaving a volume $\mathcal{V}$ is

$$\oint_S \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) d\tau.$$

Because charge is conserved, whatever flows out through the surface must come at the expense of what remains inside:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) d\tau = -\frac{d}{dt} \int_{\mathcal{V}} \rho d\tau = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} \right) d\tau.$$

(The minus sign reflects the fact that an *outward* flow *decreases* the charge left in $\mathcal{V}$.) Since this applies to *any* volume, we conclude that

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.} \quad (5.29)$$

This is the precise mathematical statement of local charge conservation; it is called the **continuity equation**.

For future reference, let me summarize the “dictionary” we have implicitly developed for translating equations into the forms appropriate to point, line, surface, and volume currents:

$$\sum_{i=1}^n () q_i \mathbf{v}_i \sim \int_{\text{line}} () \mathbf{I} dl \sim \int_{\text{surface}} () \mathbf{K} da \sim \int_{\text{volume}} () \mathbf{J} d\tau. \quad (5.30)$$

This correspondence, which is analogous to $q \sim \lambda dl \sim \sigma da \sim \rho d\tau$ for the various charge distributions, generates Eqs. 5.15, 5.24, and 5.27 from the original Lorentz force law (5.1).

Problem 5.4 Suppose that the magnetic field in some region has the form

$$\mathbf{B} = kz \hat{\mathbf{x}}$$

(where k is a constant). Find the force on a square loop (side a), lying in the yz plane and centered at the origin, if it carries a current I , flowing counterclockwise, when you look down the x axis.

Problem 5.5 A current I flows down a wire of radius a .

- If it is uniformly distributed over the surface, what is the surface current density K ?
- If it is distributed in such a way that the volume current density is inversely proportional to the distance from the axis, what is $J(s)$?